

INVERSE NODAL PROBLEMS FOR DIRAC-TYPE INTEGRO-DIFFERENTIAL OPERATORS

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ABSTRACT. The inverse nodal problem for Dirac differential operator perturbed by a Volterra integral operator is studied. We prove that dense subset of the nodal points determines the coefficients of differential and integral part of the operator. We also provide a uniqueness theorem and an algorithm to reconstruct the coefficients of the problem by using the nodal points.

1. Introduction

We consider the boundary value problem L generated by the system of Dirac integro-differential equations:

$$(1) \quad \ell[Y(x)] := BY'(x) + \Omega(x)Y(x) + \int_0^x M(x, t)Y(t)dt = \lambda Y(x), \quad x \in (0, \pi),$$

subject to the boundary conditions

$$(2) \quad U(y) : = y_1(0) \sin \alpha + y_2(0) \cos \alpha = 0$$

$$(3) \quad V(y) : = y_1(\pi) \sin \beta + y_2(\pi) \cos \beta = 0$$

where α, β are real constants and λ is the spectral parameter, $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\Omega(x) = \begin{pmatrix} V(x) + m & 0 \\ 0 & V(x) - m \end{pmatrix}$, $M(x, t) = \begin{pmatrix} M_{11}(x, t) & M_{12}(x, t) \\ M_{21}(x, t) & M_{22}(x, t) \end{pmatrix}$, $Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $\Omega(x)$ and $M(x, t)$ are real-valued functions in the class of $W_2^1(0, \pi)$, where m is a real constant. Throughout this paper, we denote $p(x) = V(x) + m$, $r(x) = V(x) - m$.

In 1988, the first results of the inverse nodal Sturm–Liouville problem was given by McLaughlin [13] who proved that the potential of the Sturm–Liouville problem can be determined by a given dense subset of nodal points of the eigenfunctions. In 1989, Hald and McLaughlin consider more general boundary conditions and give some numerical schemes for the reconstruction of the potential from nodal points [9]. Yang provided an algorithm to solve inverse nodal Sturm–Liouville problem in 1997 [17]. Inverse nodal problems for Sturm–Liouville or diffusion operators have been studied in the several papers ([1], [2], [3], [6], [14], [15], [16], [19] and [20]). The inverse nodal problems for Dirac operators with various boundary conditions

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have been solved in [8], [18] and [21]. In their works, it was shown that the zeros of the first components of the eigenfunctions determines the coefficients of operator.

Nowadays, the studies concerning the perturbation of a differential operator by a Volterra integral operator, namely the integro-differential operator, are beginning to have a significant place in the literature ([4], [5], [7], [11] and [12]). For Sturm-Liouville type integro-differential operators, there exist some studies about inverse problems but there is no study for Dirac type integro-differential operators. The inverse nodal problem for Sturm-Liouville type integro-differential operators was first studied by [10]. In their study, it is shown that the potential function can be determined by using nodal points while the coefficient of the integral operator is known. In our study, we prove that the integral operator can be partially determined as well as the potential function and the other coefficients of the problem.

2. Preliminaries

Let $\varphi(x, \lambda) = (\varphi_1(x, \lambda), \varphi_2(x, \lambda))^T$ be the solution of (1) satisfying the initial condition $\varphi(0, \lambda) = (\cos \alpha, -\sin \alpha)^T$. For each fixed x and t , this solution is an entire function of λ .

It is clear that $\varphi(x, \lambda)$ satisfies the following integral equations:

$$\begin{aligned}
 (4) \quad \varphi_1(x, \lambda) &= \cos(\lambda x - \alpha) + \int_0^x \sin \lambda(x-t)p(t)\varphi_1(t, \lambda)dt \\
 &+ \int_0^x \cos \lambda(x-t)r(t)\varphi_2(t, \lambda)dt \\
 &+ \int_0^x \int_0^t \sin \lambda(x-t) \{M_{11}(t, \xi)\varphi_1(\lambda, \xi) + M_{12}(t, \xi)\varphi_2(\lambda, \xi)\} d\xi dt \\
 &+ \int_0^x \int_0^t \cos \lambda(x-t) \{M_{21}(t, \xi)\varphi_1(\lambda, \xi) + M_{22}(t, \xi)\varphi_2(\lambda, \xi)\} d\xi dt
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad \varphi_2(x, \lambda) &= \sin(\lambda x - \alpha) - \int_0^x \cos \lambda(x-t)p(t)\varphi_1(t, \lambda)dt \\
 &+ \int_0^x \sin \lambda(x-t)r(t)\varphi_2(t, \lambda)dt \\
 &- \int_0^x \int_0^t \cos \lambda(x-t) \{M_{11}(t, \xi)\varphi_1(\lambda, \xi) + M_{12}(t, \xi)\varphi_2(\lambda, \xi)\} d\xi dt \\
 &+ \int_0^x \int_0^t \sin \lambda(x-t) \{M_{21}(t, \xi)\varphi_1(\lambda, \xi) + M_{22}(t, \xi)\varphi_2(\lambda, \xi)\} d\xi dt
 \end{aligned}$$

To apply the method of successive approximations to the system (4) and (5), we denote

$$\begin{aligned}
 \varphi_{1,0}(x, \lambda) &= \cos(\lambda x - \alpha), \\
 \varphi_{1,n+1}(x, \lambda) &= \int_0^x \sin \lambda(x-t)p(t)\varphi_{1,n}(t, \lambda)dt \\
 &+ \int_0^x \cos \lambda(x-t)r(t)\varphi_{2,n}(t, \lambda)dt \\
 &+ \int_0^x \int_0^t \sin \lambda(x-t) \{M_{11}(t, \xi)\varphi_{1,n}(\lambda, \xi) + M_{12}(t, \xi)\varphi_{2,n}(\lambda, \xi)\} d\xi dt \\
 &+ \int_0^x \int_0^t \cos \lambda(x-t) \{M_{21}(t, \xi)\varphi_{1,n}(\lambda, \xi) + M_{22}(t, \xi)\varphi_{2,n}(\lambda, \xi)\} d\xi dt
 \end{aligned}$$

and

$$\begin{aligned}
\varphi_{2,0}(x, \lambda) &= \sin(\lambda x - \alpha), \\
\varphi_{2,n+1}(x, \lambda) &= - \int_0^x \cos \lambda(x-t) p(t) \varphi_{1,n}(t, \lambda) dt \\
&\quad + \int_0^x \sin \lambda(x-t) r(t) \varphi_{2,n}(t, \lambda) dt \\
&\quad - \int_0^x \int_0^t \cos \lambda(x-t) \{ M_{11}(t, \xi) \varphi_{1,n}(\lambda, \xi) + M_{12}(t, \xi) \varphi_{2,n}(\lambda, \xi) \} d\xi dt \\
&\quad + \int_0^x \int_0^t \sin \lambda(x-t) \{ M_{21}(t, \xi) \varphi_{1,n}(\lambda, \xi) + M_{22}(t, \xi) \varphi_{2,n}(\lambda, \xi) \} d\xi dt.
\end{aligned}$$

Then we have

$$\begin{aligned}
\varphi_{1,1}(x, \lambda) &= \omega(x) \sin(\lambda x - \alpha) + \frac{m \sin \alpha}{\lambda} \sin \lambda x \\
&\quad - \frac{K(x)}{2\lambda} \cos(\lambda x - \alpha) - \frac{L(x)}{2\lambda} \sin(\lambda x - \alpha) + o\left(\frac{e^{|\tau|x}}{\lambda}\right), \\
\varphi_{2,1}(x, \lambda) &= -\omega(x) \cos(\lambda x - \alpha) - \frac{m \cos \alpha}{\lambda} \sin \lambda x \\
&\quad - \frac{K(x)}{2\lambda} \sin(\lambda x - \alpha) + \frac{L(x)}{2\lambda} \cos(\lambda x - \alpha) + o\left(\frac{e^{|\tau|x}}{\lambda}\right),
\end{aligned}$$

and for $n \geq 1$

$$\begin{aligned}
\varphi_{1,2n+1}(x, \lambda) &= (-1)^n \frac{\omega^{2n+1}(x)}{(2n+1)!} \sin(\lambda x - \alpha) + (-1)^n \frac{\omega^{2n}(x)}{(2n)!} \frac{m \sin \alpha}{\lambda} \sin \lambda x \\
&\quad + (-1)^n \frac{\omega^{2n-1}(x)}{(2n-1)!} \frac{m^2 x}{2\lambda} \cos(\lambda x - \alpha) - (-1)^n \frac{\omega^{2n}(x)}{(2n)!} \frac{K(x)}{2\lambda} \cos(\lambda x - \alpha) \\
&\quad - (-1)^n \frac{\omega^{2n}(x)}{(2n)!} \frac{L(x)}{2\lambda} \sin(\lambda x - \alpha) + o\left(\frac{e^{|\tau|x}}{\lambda}\right),
\end{aligned}$$

$$\begin{aligned}
\varphi_{1,2n}(x, \lambda) &= (-1)^n \frac{\omega^{2n}(x)}{(2n)!} \cos(\lambda x - \alpha) + (-1)^n \frac{\omega^{2n-1}(x)}{(2n-1)!} \frac{m \sin \alpha}{\lambda} \cos \lambda x \\
&\quad - (-1)^n \frac{\omega^{2n-2}(x)}{(2n-2)!} \frac{m^2 x}{2\lambda} \sin(\lambda x - \alpha) + (-1)^n \frac{\omega^{2n-1}(x)}{(2n-1)!} \frac{K(x)}{2\lambda} \sin(\lambda x - \alpha) \\
&\quad - (-1)^n \frac{\omega^{2n-1}(x)}{(2n-1)!} \frac{L(x)}{2\lambda} \cos(\lambda x - \alpha) + o\left(\frac{e^{|\tau|x}}{\lambda}\right),
\end{aligned}$$

$$\begin{aligned}
\varphi_{2,2n+1}(x, \lambda) &= -(-1)^n \frac{\omega^{2n+1}(x)}{(2n+1)!} \cos(\lambda x - \alpha) - (-1)^n \frac{\omega^{2n}(x)}{(2n)!} \frac{m \cos \alpha}{\lambda} \sin \lambda x \\
&\quad + (-1)^n \frac{\omega^{2n-1}(x)}{(2n-1)!} \frac{m^2 x}{2\lambda} \sin(\lambda x - \alpha) - (-1)^n \frac{\omega^{2n}(x)}{(2n)!} \frac{K(x)}{2\lambda} \sin(\lambda x - \alpha) \\
&\quad + (-1)^n \frac{\omega^{2n}(x)}{(2n)!} \frac{L(x)}{2\lambda} \cos(\lambda x - \alpha) + o\left(\frac{e^{|\tau|x}}{\lambda}\right),
\end{aligned}$$

$$\begin{aligned}
\varphi_{2,2n}(x, \lambda) &= (-1)^n \frac{\omega^{2n}(x)}{(2n)!} \sin(\lambda x - \alpha) - (-1)^n \frac{\omega^{2n-1}(x)}{(2n-1)!} \frac{m \cos \alpha}{\lambda} \cos \lambda x \\
&\quad + (-1)^n \frac{\omega^{2n-2}(x)}{(2n-2)!} \frac{m^2 x}{2\lambda} \cos(\lambda x - \alpha) - (-1)^n \frac{\omega^{2n-1}(x)}{(2n-1)!} \frac{K(x)}{2\lambda} \cos(\lambda x - \alpha) \\
&\quad - (-1)^n \frac{\omega^{2n-1}(x)}{(2n-1)!} \frac{L(x)}{2\lambda} \sin(\lambda x - \alpha) + o\left(\frac{e^{|\tau|x}}{\lambda}\right),
\end{aligned}$$

where, $\omega(x) = \frac{1}{2} \int_0^x (p(t) + r(t)) dt = \int_0^x V(t) dt$, $K(x) = \int_0^x (M_{11}(t, t) + M_{22}(t, t)) dt$, $L(x) = \int_0^x (M_{12}(t, t) - M_{21}(t, t)) dt$ and $\tau = \text{Im } \lambda$.

Thus, the functions $\varphi_1(x, \lambda)$ and $\varphi_2(x, \lambda)$ have the following asymptotic formulae:

$$\begin{aligned}
(6) \quad \varphi_1(x, \lambda) &= \cos(\lambda x - \omega(x) - \alpha) + \frac{m \sin \alpha}{\lambda} \sin(\lambda x - \omega(x)) \\
&\quad + \frac{m^2 x}{2\lambda} \sin(\lambda x - \omega(x) - \alpha) - \frac{K(x)}{2\lambda} \cos(\lambda x - \omega(x) - \alpha) \\
&\quad - \frac{L(x)}{2\lambda} \sin(\lambda x - \omega(x) - \alpha) + o\left(\frac{e^{|\tau|x}}{\lambda}\right),
\end{aligned}$$

$$\begin{aligned}
(7) \quad \varphi_2(x, \lambda) &= \sin(\lambda x - \omega(x) - \alpha) - \frac{m \cos \alpha}{\lambda} \sin(\lambda x - \omega(x)) \\
&\quad - \frac{m^2 x}{2\lambda} \cos(\lambda x - \omega(x) - \alpha) - \frac{K(x)}{2\lambda} \sin(\lambda x - \omega(x) - \alpha) \\
&\quad + \frac{L(x)}{2\lambda} \cos(\lambda x - \omega(x) - \alpha) + o\left(\frac{e^{|\tau|x}}{\lambda}\right)
\end{aligned}$$

for sufficiently large $|\lambda|$, uniformly in x .

The characteristic function $\Delta(\lambda)$ of the problem (1)-(3) is defined by the relation

$$(8) \quad \Delta(\lambda) = \varphi_1(\pi, \lambda) \sin \beta + \varphi_2(\pi, \lambda) \cos \beta,$$

It is obvious that $\Delta(\lambda)$ is an entire function and its zeros, namely $\{\lambda_n\}_{n \geq 0}$, coincide with the eigenvalues of the problem (1)-(3). Using the asymptotic formulae (6) and (7), one can easily obtain

$$\begin{aligned}
(9) \quad \Delta(\lambda) &= \sin(\lambda\pi - \omega(\pi) + \beta - \alpha) - \frac{m^2 \pi}{2\lambda} \cos(\lambda\pi - \omega(\pi) + \beta - \alpha) \\
&\quad - \frac{K(\pi)}{2\lambda} \sin(\lambda\pi - \omega(\pi) + \beta - \alpha) + \frac{L(\pi)}{2\lambda} \cos(\lambda\pi - \omega(\pi) + \beta - \alpha) \\
&\quad - \frac{m}{\lambda} \sin(\lambda\pi - \omega(\pi)) \cos(\beta + \alpha) + o\left(\frac{e^{|\tau|\pi}}{\lambda}\right)
\end{aligned}$$

for sufficiently large $|\lambda|$. Since the eigenvalues of the problem (1)-(3) are the roots

of $\Delta(\lambda_n) = 0$, we can write the following equation for them:

$$\begin{aligned}
&\left(1 - \frac{K(\pi)}{2\lambda_n} - \frac{m}{\lambda_n} \cos(\beta + \alpha) \cos(\alpha - \beta)\right) \tan(\lambda_n \pi - \omega(\pi) + \beta - \alpha) \\
&= \frac{m^2 \pi}{2\lambda_n} - \frac{L(\pi)}{2\lambda_n} + \frac{m}{\lambda_n} \cos(\beta + \alpha) \sin(\alpha - \beta) + o\left(\frac{e^{|\tau|\pi}}{\lambda_n}\right)
\end{aligned}$$

which implies that

$$\begin{aligned} \tan(\lambda_n \pi - \omega(\pi) + \beta - \alpha) &= \left(\frac{m^2 \pi}{2\lambda_n} - \frac{L(\pi)}{2\lambda_n} + \frac{m}{\lambda_n} \cos(\beta + \alpha) \sin(\alpha - \beta) \right) \times \\ &\times \left(1 + \frac{K(\pi)}{2\lambda_n} + \frac{m}{\lambda_n} \cos(\beta + \alpha) \cos(\alpha - \beta) + o\left(\frac{1}{\lambda_n}\right) \right) \end{aligned}$$

for sufficiently large n . We obtain from the last equation,

$$\begin{aligned} (10) \quad \lambda_n &= n + \frac{1}{\pi} \int_0^\pi V(t) dt + \frac{\alpha - \beta}{\pi} \\ &+ \frac{1}{2n\pi} (m^2 \pi - L(\pi) + 2m \cos(\beta + \alpha) \sin(\alpha - \beta)) + o\left(\frac{1}{n}\right) \end{aligned}$$

for $n = 0, \pm 1, \pm 2, \dots$

3. Main Results

Lemma 1. *For sufficiently large n , the first component $\varphi_1(x, \lambda_n)$ of the eigenfunction $\varphi(x, \lambda_n)$ has exactly n nodes $\{x_n^j : j = 0, n\}$ in the interval $(0, \pi)$: $0 < x_n^0 < x_n^1 < \dots < x_n^n < \pi$. The numbers $\{x_n^j\}$ satisfy the following asymptotic formula:*

$$\begin{aligned} (11) \quad x_n^j &= \frac{(j + 1/2)\pi}{n} + \frac{\omega(x_n^j) + \alpha}{n} \\ &- \frac{(j + 1/2)\pi}{n} \left(\frac{\omega(\pi) + \alpha - \beta}{n\pi} \right) - \frac{\omega(\pi) + \alpha - \beta}{n^2\pi} (\omega(x_n^j) + \alpha) \\ &+ \frac{1}{2n^2} (m^2 x_n^j - L(x_n^j) + m \sin 2\alpha) + o\left(\frac{1}{n^2}\right). \end{aligned}$$

Proof. From (6), the following asymptotic formula can be written for sufficiently large n .

$$\begin{aligned} \varphi_1(x, \lambda_n) &= \cos(\lambda_n x - \omega(x) - \alpha) + \frac{m \sin 2\alpha}{2\lambda_n} \sin(\lambda_n x - \omega(x) - \alpha) \\ &+ \frac{m \sin^2 \alpha}{\lambda_n} \cos(\lambda_n x - \omega(x) - \alpha) + \frac{m^2 x}{2\lambda_n} \sin(\lambda_n x - \omega(x) - \alpha) \\ &- \frac{K(x)}{2\lambda_n} \cos(\lambda_n x - \omega(x) - \alpha) - \frac{L(x)}{2\lambda_n} \sin(\lambda_n x - \omega(x) - \alpha) + o\left(\frac{e^{|\tau_n| x}}{\lambda_n}\right) \end{aligned}$$

From $\varphi_1(x_n^j, \lambda_n) = 0$, we get

$$\begin{aligned} \cos(\lambda_n x_n^j - \omega(x_n^j) - \alpha) &= \frac{m \sin 2\alpha}{2\lambda_n} \sin(\lambda_n x_n^j - \omega(x_n^j) - \alpha) \\ &+ \frac{m \sin^2 \alpha}{\lambda_n} \cos(\lambda_n x_n^j - \omega(x_n^j) - \alpha) \\ &+ \frac{m^2 x_n^j}{2\lambda_n} \sin(\lambda_n x_n^j - \omega(x_n^j) - \alpha) \end{aligned}$$

$$\begin{aligned}
& -\frac{K(x_n^j)}{2\lambda_n} \cos(\lambda_n x_n^j - \omega(x_n^j) - \alpha) \\
& -\frac{L(x_n^j)}{2\lambda_n} \sin(\lambda_n x_n^j - \omega(x_n^j) - \alpha) + o\left(\frac{e^{|\tau_n|x}}{\lambda_n}\right).
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\tan\left(\lambda_n x_n^j - \omega(x_n^j) - \alpha - \frac{\pi}{2}\right) &= \left(1 - \frac{K(x)}{2\lambda_n} + \frac{m \sin^2 \alpha}{\lambda_n}\right)^{-1} \times \\
&\times \left(\frac{m^2 x_n^j}{2\lambda_n} - \frac{L(x_n^j)}{2\lambda_n} + \frac{m \sin 2\alpha}{\lambda_n} + o\left(\frac{1}{\lambda_n}\right)\right).
\end{aligned}$$

Taking into account Taylor's expansion formula for the arctangent, we get

$$\lambda_n x_n^j - \omega(x_n^j) - \alpha - \frac{\pi}{2} = j\pi + \frac{1}{2\lambda_n} (m^2 x_n^j - L(x_n^j) + m \sin 2\alpha) + o\left(\frac{1}{\lambda_n}\right).$$

It follows from the last equality

$$x_n^j = \frac{(j + \frac{1}{2})\pi + \omega(x_n^j) + \alpha}{\lambda_n} + \frac{1}{2\lambda_n^2} (m^2 x_n^j - L(x_n^j) + m \sin 2\alpha) + o\left(\frac{1}{\lambda_n^2}\right).$$

The relation (11) is proven by using the asymptotic formula

$$\lambda_n^{-1} = \frac{1}{n} \left\{ 1 - \frac{\omega(\pi) + \alpha - \beta}{n\pi} - \frac{(m^2\pi - L(\pi) + 2m \cos(\beta + \alpha) \sin(\alpha - \beta))}{2n^2\pi} + o\left(\frac{1}{n^2}\right) \right\}$$

From (11), we have the following asymptotic expression for nodal lengths:

$$l_n^j := x_n^{j+1} - x_n^j = \frac{\pi}{n} + o\left(\frac{1}{n}\right).$$

One can easily see that $\varphi_1(\frac{k\pi}{n}, \lambda_n)$ and $\varphi_1(\frac{(k+1)\pi}{n}, \lambda_n)$ have different signs for each fixed k and for sufficiently large n . Thus, the function $\varphi_1(x, \lambda_n)$ has exactly n nodes in $(0, \pi)$. \square

Let X be the set of nodal points and $\omega(\pi) = 0$. For each fixed $x \in (0, \pi)$ we can choose a sequence $(x_n^j) \subset X$ so that x_n^j converges to x . Then the following limits are exist and finite:

$$(12) \quad \lim_{n \rightarrow \infty} n \left(x_n^{j(n)} - \frac{(j + \frac{1}{2})\pi}{n} \right) = f(x),$$

where

$$f(x) = \omega(x) + \alpha - \frac{x}{\pi} (\alpha - \beta)$$

and

$$(13) \quad \lim_{n \rightarrow \infty} 2n^2 \left(x_n^j - \frac{(j + \frac{1}{2})\pi - \omega(x_n^j)}{n} + \frac{(j + 1/2)\pi}{n} \left(\frac{\alpha - \beta}{n\pi} \right) \right) = g(x),$$

where

$$g(x) = -L(x) + 2\frac{\beta - \alpha}{\pi} (\omega(x) + \alpha) + m^2 x + m \sin 2\alpha$$

Therefore, proof of the following theorem is clear.

Theorem 1. *The given dense subset of nodal points X uniquely determines the potential $V(x)$, the function $L'(x) = M_{12}(x, x) - M_{21}(x, x)$ a.e. on $(0, \pi)$, and the coefficients α and β of the boundary conditions. Moreover, $V(x)$, $L'(x)$, α and β can be reconstructed by the following formulae:*

Step-1: *For each fixed $x \in (0, \pi)$, choose a sequence $(x_n^{j(n)}) \subset X$ such that*

$$\lim_{n \rightarrow \infty} x_n^{j(n)} = x;$$

Step-2: *Find the function $f(x)$ from (12) and calculate*

$$\begin{aligned} V(x) &= f'(x) \\ \alpha &= f(0) \\ \beta &= f(\pi) \end{aligned}$$

Step-3: *If $\sin 2\alpha \neq 0$, find the function $g(x)$ from (13) and calculate*

$$\begin{aligned} m &= \frac{g(0) + 2\alpha(\alpha - \beta)}{\sin 2\alpha} \\ L'(x) &= -g'(x) + 2\frac{\beta - \alpha}{\pi}V(x) + m^2 \end{aligned}$$

otherwise assume m is known.

Example 1. *Let $\{x_n^j\} \subset X$ be the dense subset of nodal points in $(0, \pi)$ given by the following asymptotics:*

$$\begin{aligned} x_n^j &= \frac{(j + 1/2)\pi}{n} + \frac{\frac{\pi}{4} + \sin \frac{(j + 1/2)\pi}{n}}{n} \\ &+ \frac{1}{2n^2} \left(\frac{(j + 1/2)\pi}{n} + \sin \frac{(j + 1/2)\pi}{n} + 1 \right) + o\left(\frac{1}{n^2}\right). \end{aligned}$$

It can be calculated from (12) and (13) that,

$$\begin{aligned} f(x) &= \frac{\pi}{4} + \sin x \\ g(x) &= 1 + x + \sin x \end{aligned}$$

Therefore, it is obtained by using the algorithm in Theorem 1,

$$\begin{aligned} V(x) &= f'(x) = \cos x, \\ \alpha &= f(0) = \frac{\pi}{4} = f(\pi) = \beta, \\ M_{12}(x, x) - M_{21}(x, x) &= L'(x) = -\cos x \\ m &= 1. \end{aligned}$$

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